On a conjecture of Deutsch, Sagan, and Wilson

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Abstract

We prove a recent conjecture due to Deutsch, Sagan, and Wilson stating that the finite sequence obtained from the first p central trinomial coefficients modulo p by replacing nonzero terms by 1's is palindromic, for any prime number $p \geq 5$.

1 Introduction

In the recent paper [3] Deutsch and Sagan study several combinatorial sequences reduced modulo prime numbers. They are in particular interested in the values modulo p of the central trinomial coefficients. Let us recall that the nth central trinomial coefficient is defined as the largest coefficient in the expansion of the polynomial $(1+x+x^2)^n$. Deutsch and Sagan make the following conjecture [3, Conjecture 5.8] (also stated by Wilson, see [3]): for each prime $p \geq 5$ and for each $j < p$, the number T_j is divisible by p if and only if T_{p-1-j} is divisible by p. We give here an elementary proof of this conjecture.

2 The central trinomial coefficients modulo a prime

In this section we first recall a classical result (see [6, sequence A002426] for example).

Proposition 1 *The generating function F of the central trinomial coeficient satisfies:*

$$F(x) := \sum_{n \geq 0} T_n x^n = \frac{1}{\sqrt{1-2x-3x^2}}.$$

Remark 1 Note that, using [1, Theorem 6.4] (see also [2]), this implies that the sequence $(T_n)_{n\geq 0}$ satisfies: if p is prime and if the base p expansion of n is $n = \sum n_j p^j$, then $T_n \equiv \prod T_{n_j} \bmod p$, which is Theorem 4.7 of [3].

An easy consequence of Proposition 1 is the following statement.

Proposition 2 *Let p be an odd prime. Then we have the following identity*

$$\sum_{0\le n\le p-1} T_n x^n \equiv (1-2x-3x^2)^{\frac{p-1}{2}} \bmod p.$$

Proof. From Proposition 1 we have

$$(1-2x-3x^2)^{\frac{p-1}{2}} \left(\sum_{n\ge 0} T_n x^n\right)^{p-1} \equiv 1 \bmod p. \tag{1}$$

On the other hand, using the "p-Lucas property" recalled in Remark 1 above, we have

$$\begin{aligned}
\sum_{n\ge 0} T_n x^n &\equiv \sum_{0\le j\le p-1}\sum_{n\ge 0} T_{pn+j}x^{pn+j} &&\equiv \sum_{0\le j\le p-1}\sum_{n\ge 0} T_n T_j x^{pn} x^j \\
&\equiv \sum_{0\le j\le p-1} T_j x^j \sum_{n\ge 0} T_n x^{pn} &&\equiv \left(\sum_{0\le j\le p-1} T_j x^j\right)\left(\sum_{n\ge 0} T_n x^n\right)^p \bmod p
\end{aligned}$$

which yields

$$\left(\sum_{0\le j\le p-1} T_j x^j\right)\left(\sum_{n\ge 0} T_n x^n\right)^{p-1} \equiv 1 \bmod p. \tag{2}$$

Comparing Equations 1 and 2 finishes the proof. $\square$

3 Proof of the conjecture

We first prove a proposition on the nonzero coefficients of a quadratic polynomial raised to an integer power.

Proposition 3 *Let $1+ax+bx^2$ be a polynomial with coefficients in a commutative field K, with $b \neq 0$. Let k be a positive integer. Then, noting $(1+ax+bx^2)^k := \sum_{0\le j\le 2k} \alpha_j x^j$, we have $\alpha_j = 0$ if and only if $\alpha_{2k-j} = 0$.*

Proof. We write

$$\sum_{0\le j\le 2k} \alpha_j b^{k-j} x^{2k-j} = b^k x^{2k} \sum_{0\le j\le 2k} \alpha_j \left(\frac{1}{bx}\right)^j = b^k x^{2k}\left(1+\frac{a}{bx}+\frac{b}{(bx)^2}\right)^k = (bx^2+ax+1)^k.$$

But the sum on the left can also be written $\sum_{0\le j\le 2k} \alpha_{2k-j} b^{j-k} x^j$; thus, for all $j \in [0, 2k]$, we have $\alpha_{2k-j} = b^{k-j}\alpha_j$ which implies our claim. $\square$

As an immediate corollary, we get a proof of the conjecture of Deutsch, Sagan, and Wilson.

Theorem *For any prime $p \ge 5$, for any $j \in [0, p-1]$, the sequence of central trinomial coefficients $(T_n)_{n\ge 0}$ satisfies*

$$p \mid T_j \text{ if and only if } p \mid T_{p-1-j}.$$

Proof. Apply Proposition 3 with $K := \mathbb{Z}/p\mathbb{Z}$ (the finite field with p elements) and $1 + ax + bx^2 := 1 - 2x - 3x^2$, and use Proposition 2. $\square$

Remark 2 The reader can check that the proof of the Theorem above readily generalizes to proving the following. (Hint: use [2, Theorem 2 and its proof].)
Let $(R_n)_{n\geq 0}$ be a sequence of integers, such that there exists a polynomial of degree 2 with integer coefficients $P(x) := 1 + ax + bx^2$ such that $\sum_{n\geq 0} R_n x^n = (P(x))^{-1/2}$. Then, for all primes p such that p does not divide $3R_1^2 - 2R_2$ and for all $j \in [0, p-1]$, we have

$$p \mid R_j \text{ if and only if } p \mid R_{p-1-j}.$$

In particular if $(R_n)_{n\geq 0}$ is the sequence of central Delannoy numbers (see [6, sequence A001850]), then for all primes p and for all $j \in [0, p-1]$, we have

$$p \mid R_j \text{ if and only if } p \mid R_{p-1-j}.$$

Note that the p-Lucas property for this sequence is a consequence of [1, Theorem 6.4] (see also [2]) and of the fact that the generating function for the central Delannoy numbers is equal to $(1 - 6x + x^2)^{-1/2}$ (see [6, sequence A001850] for example); it is also proven in [3] and in [4]. A nice paper on sequences having the p-Lucas property is [5].

Addendum: the result was proved before almost in the same way by Tony D. Noe: On the Divisibility of Generalized Central Trinomial Coefficients, Journal of Integer Sequences, Vol. 9 (2006), Article 06.2.7
`http://www.cs.uwaterloo.ca/journals/JIS/VOL9/Noe/noe35.html`

References

[1] J.-P. Allouche, Transcendence of formal power series with rational coefficients, *Theoret. Comput. Sci.* **218** (1999) 143–160.

[2] J.-P. Allouche, D. Gouyou-Beauchamps, G. Skordev, Transcendence of binomial and Lucas' formal power series, *J. Algebra* **210** (1998) 577–592.

[3] E. Deutsch, B. E. Sagan, Congruences for Catalan and Motzkin numbers and related sequences, *J. Number Theory*, to appear.

[4] S.-P. Eu, S.-C. Liu, Y.-N. Yeh, On the congruences of some combinatorial numbers, *Stud. Appl. Math.* **116** (2006) 135–144.

[5] R. McIntosh, A generalization of a congruential property of Lucas, *Amer. Math. Monthly* **99** (1992) 231–238.

[6] N. J. A. Sloane, On-line Encyclopedia of Integer Sequences, published electronically at (`http://www.research.att.com/~njas/sequences/`).